STABILITY OF A CANONICAL SYSTEM WITH TWO DEGREES OF FREEDOM IN THE PRESENCE OF RESONANCE

PMM Vol. 32, №4, 1968, pp. 738-744

A. P. MARKEEV
(Moscow)

(Received December 27, 1967)

We investigate the problem of stability of equilibrium of a canonical system with two degrees of freedom for the case of resonance, and use the results obtained to study the stability of the steady motions of a satellite.

1. Statement of the problem. We consider a self-contained canonical system with two degrees of freedom

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad (i = 1, 2) \tag{1.1}$$

Let us assume that the coordinate origin coincides with the state of equilibrium of the system and, that the Hamiltonian H is an analytic function of generalized coordinates and impulses q_i and p_i which can be expanded into the series

$$H = H_2 + H_3 + H_4 + \ldots + H_m + \cdots \tag{1.2}$$

where H_m is a homogeneous function of the mth degree in q_i and p_i.

If H_2 is sign definite, then by the Liapunov theorem the equilibrium is stable [1]. Let us assume that H_2 is not sign definite, but the system is stable in the first approximation. Then, imposing certain restrictions on the frequencies ω_1 and ω_2 of the linear system and on the coefficients of the forms H_3 and H_4, we can solve the problem of stability of the complete system (1.1) using the theorem given in [2]. A substantial limitation of this theorem is the requirement of the absence of resonance: that for the frequencies ω_1 and ω_2 the inequalities

$$k_1\omega_1 + k_2\omega_2 \neq 0 \tag{1.3}$$

where k_i are integers satisfying the condition $0 < |k_1| + |k_2| \leqslant 4$, should hold. Some cases are, however, known [3] where the system (1.1) can become unstable in the presence of resonance.

The cases when one of the frequencies is equal to zero, or when both frequencies are equal to each other, usally correspond to the boundary of stability of the linear system and shall not be considered further. Putting $\omega_1 > \omega_2 > 0$, we find, that the inequalities (1.3) are invalid for $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$. The aim of this work is to investigate the equilibrium stability of (1.1) in these two resonant cases. We use the Birkhoff [4] transformation to obtain the Hamiltonian in the form showing the resonant character of the problem. As far as stability is concerned, both the initial and the transformed system are equivalent. The results obtained are expressed in the terms of the coefficients of the Hamiltonian (1.2) written in the form, in which its quadratic part H_2 corresponds to the normal oscillations.

2. Investigation of the stability when $\omega_1 = 2\omega_2$. Simple transformations (see Section 5) yield the Hamiltonian of the problem in the form (2.1)

$$H = i\omega_1 q_1' p_1' + i\omega_2 q_2' p_2' + \sum_{\nu=3} g'_{\nu_1\nu_2\nu_3\nu_4} q_1'^{\nu_1} q_2'^{\nu_2} p_1'^{\nu_3} p_2'^{\nu_4} + \sum_{\nu=4} h'_{\nu_1\nu_2\nu_3\nu_4} q_1'^{\nu_1} q_2'^{\nu_2} p_1'^{\nu_3} p_2'^{\nu_4} + O(|q|^5)$$

where

$$|q| = \sqrt{q_1^2 + q_2^2 + p_1^2 + p_2^2}, \qquad \nu = \nu_1 + \nu_2 + \nu_3 + \nu_4$$

We shall try to remove the third order terms from (1.1), applying the Birkhoff transformation. We find that when $\omega_1 = 2\omega_2$, then all terms except the resonant ones can be made to vanish and the Hamiltonian will become, in the new variables q_i'' and p_i'',

$$H = i\omega_1 q_1'' p_1'' + i\omega_2 q_2'' p_2'' + g'_{1002} q_1'' p_2''^2 + g'_{0210} q_2''^2 p_1'' + O(|q|^4) \tag{2.2}$$

where (see Section 5)

$$g'_{1002} = x_{1002} + iy_{1002}, \quad g_{0210}' = -{}^1/_2\omega_1^{-1}\omega_2^2(y_{1002} + ix_{1002})$$

After the canonical change of variables

$$q_1'' = \omega_1^{-1/2}(q_1^\circ - ip_1^\circ), \qquad q_2'' = \omega_2^{-1/2}(iq_2^\circ - p_2^\circ),$$
$$p_1'' = {}^1/_2\omega_1^{1/2}(-iq_1^\circ + p_1^\circ), \qquad p_2'' = {}^1/_2\omega_2^{1/2}(q_2^\circ - ip_2^\circ) \tag{2.3}$$

the Hamiltonian becomes (2.4)

$$H = {}^1/_2\omega_1(q_1^{\circ 2} + p_1^{\circ 2}) - {}^1/_2\omega_2(q_2^{\circ 2} + p_2^{\circ 2}) + {}^1/_2\sqrt{2\omega_2}\,[{}^1/_2(q_2^{\circ 2} - p_2^{\circ 2})(x_{1002}q_1^\circ + y_{1002}p_1^\circ) + $$
$$+ q_2^\circ p_2^\circ(y_{1002}q_1^\circ - x_{1002}p_1^\circ)] + O(|q|^4)$$

Let us assume that the inequality

$$x_{1002}^2 + y_{1002}^2 \neq 0 \tag{2.5}$$

holds. Then another canonical transformation

$$q_1^\circ = q_1^*\cos\theta - p_1^*\sin\theta, \quad q_2^\circ = q_2^*, \qquad p_1^\circ = q_1^*\sin\theta + p_1^*\cos\theta, \quad p_2^\circ = p_2^* \tag{2.6}$$

where

$$\sin\theta = y_{1002}(x_{1002}^2 + y_{1002}^2)^{-1/2}, \quad \cos\theta = x_{1002}(x_{1002}^2 + y_{1002}^2)^{-1/2}$$

will yield the Hamiltonian in the form

$$H = \omega_2(q_1^{*2} + p_1^{*2}) - {}^1/_2\omega_2(q_2^{*2} + p_2^{*2}) - {}^1/_2\sqrt{2\omega_2(x_{1002}^2 + y_{1002}^2)}\times$$
$$\times[{}^1/_2 q_1^*(p_2^{*2} - q_2^{*2}) + p_1^* q_2^* p_2^*] + O(|q|^4) \tag{2.7}$$

We shall show that, when the inequality (2.5) holds, then the equilibrium is unstable, and we shall use the Chetaev theorem [5]. In the present case the function

$$V = {}^1/_2 p_1^*(p_2^{*2} - q_2^{*2}) - q_1^* q_2^* p_2^* \tag{2.8}$$

can be used as an aid in solving the problem of instability.

We can assume, for example, that the region $V > 0$ is defined by the inequalities $q_1^* < 0,\ p_1^* < 0,\ p_2^* < q_2^* < 0$. By virtue of the equations of motion with the Hamiltonian (2.7), the derivative dV/dt will be (2.9)

$$dV/dt = {}^1/_8\sqrt{2\omega_2(x_{1002}^2 + y_{1002}^2)}\,(q_2^{*2} + p_2^{*2})[(q_2^{*2} + p_2^{*2}) + 4(q_1^{*2} + p_1^{*2})] + Q(|q|^5)$$

From (2.9) we see that in the region $V > 0$ and for sufficiently small q_i^* and p_i^*, the function dV/dt is positive definite. This proves the instability of equilibrium under the condition (2.5).

Let us now assume that (2.5) does not hold. Then the terms $g_{1002}' q_1' p_2'^2$ and $g_{0210}' q_2'^2 p_1'$ will be absent from the Hamiltonian (2.1) which, consequently, may assume the following form with the aid of the Birkhoff transformation (2.10)

$$H = i\omega_1 q_1'' p_1'' + i\omega_2 q_2'' p_2'' + l_{2020}(q_1'' p_1'')^2 + l_{1111} q_1'' p_1'' q_2'' p_2'' + l_{0202}(q_2'' p_2'')^2 + O(|q|^5)$$

where l_{2020}, l_{1111} and l_{0202} are real, and given by the formulas in Section 5. Results of [2] show that when the inequality

$$l_{2020} - 2l_{1111} + 4l_{0202} \neq 0 \tag{2.11}$$

holds, then the equilibrium is stable. This constitutes the proof of the following theorem.

Theorem 2.1. If the inequality $x_{1002}^2 + y_{1002}^2 \neq 0$ holds for the Hamiltonian of a perturbed motion, then the equilibrium is unstable. If, on the other hand, $x_{1002}^2 + y_{1002}^2 = 0$ and $l_{2020} - 2l_{1111} + 4l_{0202} \neq 0$, then the equilibrium is stable.

3. Investigation of stability when $\omega_1 = 3\omega_2$. We shall consider the case of a resonance when $\omega_1 = 3\omega_2$. Using the Birkhoff transformation we can cause all third degree terms in (2.1) to vanish. Out of the fourth degree terms, only the resonant ones and those containing q_i'' and p_i'' in the same degree will remain. Therefore the normal form of the Hamiltonian will be, in the case when $\omega_1 = 3\omega_2$,

$$\begin{aligned} H = {} & i\omega_1 q_1'' p_1'' + i\omega_2 q_2'' p_2'' + l_{2020}(q_1''p_1'')^2 + l_{1111} q_1''p_1''q_2''p_2'' + \\ & + l_{0202}(q_2''p_2'')^2 + l_{1003} q_1'' p_2''^3 + l_{0310} q_2''^3 p_1'' + O(|q|^5) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
l_{2020} = {} & h'_{2020} - \frac{3}{i\omega_1} g'_{3000}g'_{0030} - \frac{3}{i\omega_1} g'_{2010}g'_{1020} + \frac{1}{i(2\omega_1-\omega_2)} g'_{0120}g'_{2001} - \\
& - \frac{1}{i\omega_2} g'_{1110}g'_{1011} - \frac{1}{i(2\omega_1+\omega_2)} g'_{2100}g'_{0021} \\
l_{1111} = {} & h'_{1111} + \frac{4}{i(\omega_1-2\omega_2)} g'_{0210}g'_{1002} - \frac{4}{i(\omega_1+2\omega_2)} g'_{1200}g'_{0012} - \frac{4}{i(2\omega_1+\omega_2)} g'_{2100}g'_{0021} - \\
& - \frac{4}{i(2\omega_1-\omega_2)} g'_{2001}g'_{0120} - \frac{2}{i\omega_1} g'_{2010}g'_{0111} - \frac{2}{i\omega_1} g'_{1101}g'_{1020} - \frac{2}{i\omega_2} g'_{0201}g'_{1011} - \frac{2}{i\omega_2} g'_{1110}g'_{0102} \\
l_{0202} = {} & h'_{0202} - \frac{3}{i\omega_2} g'_{0300}g'_{0003} - \frac{3}{i\omega_2} g'_{0102}g'_{0201} - \frac{1}{i(\omega_1-2\omega_2)} g'_{1002}g'_{0210} - \\
& - \frac{1}{i\omega_1} g'_{1101}g'_{0111} - \frac{1}{i(\omega_1+2\omega_2)} g'_{1200}g'_{0012} \\
l_{1003} = {} & h'_{1003} - \frac{2}{i(2\omega_1-\omega_2)} g'_{2001}g'_{0012} - \frac{1}{i(\omega_1-2\omega_2)} g'_{1011}g'_{1002} + \\
& + \frac{2}{i\omega_2} g'_{1002}g'_{0102} - \frac{3}{i\omega_1} g'_{0103}g'_{1101} \\
l_{0310} = {} & h'_{0310} - \frac{2}{i(\omega_1+2\omega_2)} g'_{0120}g'_{1200} - \frac{1}{i\omega_2} g'_{1110}g'_{0210} + \\
& + \frac{2}{i(\omega_1-2\omega_2)} g'_{0210}g'_{0201} - \frac{1}{i\omega_2} g'_{0300}g'_{0111}
\end{aligned} \tag{3.2}$$

We note that l_{2020}, l_{1111} and l_{0202} are real and

$$l_{1003} = x_{1003} + iy_{1003}, \qquad l_{0310} = -{}^1/_{12}\,\omega_2^2 (x_{1003} - iy_{1003}) \tag{3.3}$$

Formulas yielding the coefficients (3.2) in terms of the coefficients of the initial Hamiltonian, are given in Section 5.

Change of variables in (2.3) yields

$$\begin{aligned} H = {} & {}^3/_2\,\omega_2 (q_1^{\circ 2} + p_1^{\circ 2}) - {}^1/_2\,\omega_2 (q_2^{\circ 2} + p_2^{\circ 2}) - {}^1/_4\, l_{2020} (q_1^{\circ 2} + p_1^{\circ 2})^2 + \\ & + {}^1/_4\, l_{1111} (q_1^{\circ 2} + p_1^{\circ 2})(q_2^{\circ 2} + p_2^{\circ 2}) - {}^1/_4\, l_{0202} (q_2^{\circ 2} + p_2^{\circ 2})^2 + \\ & + {}^1/_{12} \sqrt{3}\,\omega_2 \,[p_2^\circ (p_2^{\circ 2} - 3q_2^{\circ 2})(x_{1003}p_1^\circ - y_{1003}q_1^\circ) + \\ & + q_2^\circ (q_2^{\circ 2} - 3p_2^{\circ 2})(y_{1003}p_1^\circ + x_{1003}q_1^\circ)] + O(|q|^5) \end{aligned} \tag{3.4}$$

If $x_{1003}^2 + y_{1003}^2 = 0$, then, by [2], the equilibrium is stable provided that

$$l_{2020} - 3l_{1111} + 9l_{0202} \neq 0 \tag{3.5}$$

holds.

Now assume that $x_{1003}^2 + y_{1003}^2 \neq 0$. Then the canonical transformation (2.6) in which we have now $\sin\theta = x_{1003}(x_{1003}^2 + y_{1003}^2)^{-1/2}$, $\cos\theta = -y_{1003}(x_{1003}^2 + y_{1003}^2)^{-1/2}$ yields the Hamiltonian in the form

$$H = {}^3/_2\,\omega_2(q_1^{*2} + p_1^{*2}) - {}^1/_2\,\omega_2(q_2^{*2} + p_2^{*2}) - {}^1/_4\,l_{2020}(q_1^{*2} + p_1^{*2})^2 + \\ + {}^1/_4\,l_{1111}(q_1^{*2} + p_1^{*2})(q_2^{*2} + p_2^{*2}) - {}^1/_4\,l_{0202}(q_2^{*2} + p_2^{*2})^2 + \\ + {}^1/_{12}\,\omega_2\sqrt{3(x_{1003}^2 + y_{1003}^2)}\,[q_1^* p_2^*(p_2^{*2} - 3q_2^{*2}) - p_1^* q_2^*(q_2^{*2} - 3p_2^{*2})] + O(|q|^5) \tag{3.6}$$

Next we shall show that the equilibrium is unstable under the condition

$$3\omega_2\sqrt{x_{1003}^2 + y_{1003}^2} \geqslant |l_{2020} - 3l_{1111} + 9l_{0202}| \tag{3.7}$$

Again we use the Chetaev theorem and take V in the form $V = V_1 V_2$, where

$$V_1 = (p_2^{*2} + q_2^{*2} - 3p_1^{*2} - 3q_1^{*2})^2 - (p_2^{*2} + q_2^{*2})^k \qquad (k > 2) \\ V_2 = p_1^* p_2^*(p_2^{*2} - 3q_2^{*2}) + q_1^* q_2^*(q_2^{*2} - 3p_2^{*2}) \tag{3.8}$$

Obviously, a region $V > 0$ exists near the coordinate origin. It could, for example, be a set of points $V_1 < 0$ situated near the surface $p_2^{*2} + q_2^{*2} = 3(p_1^{*2} + q_1^{*2})$ in the region $V_2 < 0$ defined by the inequalities $p_1^* < 0$, $q_1^* > 0$, $p_2^* > \sqrt{3}q_2^* > 0$.

The parameter k in (3.8) can be chosen so as to ensure that the function dV/dt is positive and, that the region $V > 0$ lies within the region $dV/dt > 0$.

The derivative of V can be obtained as follows. Using the canonical polar coordinates

$$q_i^* = \sqrt{2r_i}\sin\varphi_i, \qquad p_i^* = \sqrt{2r_i}\cos\varphi_i \quad (i = 1,2)$$

we obtain the Hamiltonian in the form

$$H = 3\omega_2 r_1 - \omega_2 r_2 - l_{2020}r_1^2 + l_{1111}r_1 r_2 - l_{0202}r_2^2 + \\ + {}^1/_3\,\omega_2\sqrt{3(x_{1003}^2 + y_{1003}^2)}\,r_2\sqrt{r_1 r_2}\sin(\varphi_1 + 3\varphi_2) + O(|q|^5) \tag{3.9}$$

Another canonical transformation

$$P_1 = {}^1/_3\,r_2, \qquad Q_1 = \varphi_1 + 3\varphi_2, \qquad P_2 = -{}^1/_3\,r_1 + {}^1/_9\,r_2, \qquad Q_2 = -3\varphi_1$$

converts the Hamiltonian to

$$H = a_2 P_1^2 + a_1 P_1 P_2 + a_0 P_1\sqrt{P_1(P_1 - 3P_2)}\sin Q_1 - 9l_{2020}P_2^2 - 9\omega_2 P_2 + O(|q|^5) \tag{3.10}$$

where

$$a_0 = 3\omega_2\sqrt{x_{1003}^2 + y_{1003}^2}, \qquad a_1 = 3(2l_{2020} - 3l_{1111}), \qquad a_2 = -l_{2020} + 3l_{1111} - 9l_{0202}$$

The function V will then become

$$V = 12\sqrt{3}\,[(18P_2)^2 - (6P_1)^k]P_1\sqrt{P_1(P_1 - 3P_2)}\cos Q_1 \tag{3.11}$$

In the region $V_2 < 0$ ($\cos Q_1 < 0$) in the new variables) near the surface $p_2^{*2} + q_2^{*2} = 3(p_1^{*2} + q_1^{*2})$ ($P_2 = 0$ in the new variables), V assumes positive values when

$$P_2 = {}^1/_{18}A(6P_1)^{k/2} \tag{3.12}$$

where A is any number lying on the interval $(-1, 1)$.

Using now the equations of motion whose Hamiltonian is (3.10) and utilizing (3.12) we find, that, for $2 < k < 3$, dV/dt has the form

$$dV/dt = 12\sqrt{3}[k6^k a_0\cos^2 Q_1 + 12(1 - A^2)(a_0 + a_2\sin Q_1) + f(P_1)]P_1^{k+3} \tag{3.13}$$

where $f(P_1)$ can be arbitrarily small as P_1 tends to zero. Consequently, at sufficiently small distances from the coordinate origin, the sign of the derivative is determined by the sign of the expression

$$k6^k a_0 \cos^2 Q_1 + 12(1 - A^2)(a_0 + a_2 \sin Q_1) \tag{3.14}$$

from which we see that when $a_0 \geqslant |a_2|$ $dv/dt > 0$ and it does not become zero within the region $V > 0$. Therefore the function V satisfies all the requirements of the Chetaev theorem, and this, in turn, proves the instability under the condition (3.7).

Let us now suppose that (3.7) does not hold. If we truncate Expression (1.2) for the Hamiltonian so that no terms of order higher than fourth in q_i and p_i appear in it, we can show that the equilibrium is stable within this approximation.

Indeed, in this case the system possessing the Hamiltonian (3.10) has two integrals

$$P_2 = \text{const}, \qquad a_2P_1^2 + a_1P_1P_2 + a_0P_1\sqrt{P_1(P_1 - 3P_2)}\sin Q_1 = \text{const}$$

Let us consider the function

$$V = [a_2P_1^2 + a_1P_1P_2 + a_0P_1\sqrt{P_1(P_1 - 3P_2)}\sin Q_1]^2 + P_2^4 \tag{3.15}$$

Obviously $dV/dt = 0$ and it can easily be shown that when $a_0 < |a_2|$ then the function (3.15) is positive definite in the vicinity $q_i = p_i = 0$. Therefore, by the Liapunov theorem [1], the equilibrium is stable within the approximation considered and thiş constitutes the proof of the following theorem.

Theorem 3.1. If the inequalities

$$x_{1003}^2 + y_{1003}^2 \neq 0, \qquad 3\omega_2\sqrt{x^2_{1003} + y_{1003}^2} \geqslant |l_{2020} - 3l_{1111} + 9l_{0202}|,$$

hold simultaneously for the Hamiltonian of the perturbed motion, then the equilibrium is unstable; if, however, the last inequality is of the opposite sign and the Hamiltonian contains no terms of the order higher than the fourth, then the equilibrium is stable. It is also stable when

$$x_{1003}^2 + y_{1003}^2 = 0, \quad l_{2002} - 3l_{1111} + 9l_{0202} \neq 0$$

hold simultaneously.

4. Stability of the steady rotations of a satellite. As an example, we shall consider the stability of steady rotations of a satellite in the case of a resonace. We know [6] that when a dynamically symmetric satellite moves along a circular orbit in a central Newtonian gravitational field, it can assume several positions in the orbital coordinate system. Its axis of symmetry may:

1) be perpendicular to the orbital plane,
2) lie in the plane perpendicular to the radius vector,
3) lie in the plane perpendicular to the velocity vector,

and it rotates about this axis with the constant angular velocity.

Stability of the steady rotation of the type (2) has been fully investigated. The cases lacking a solution are those [7] of the resonance $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ for the motions of the type (1) and (3), when the function H_2 is not sign definite, but the motion is stable in the first approximation.

For the motion of the type (1), the Hamiltonian (1.2) has only even order terms (m is even). Therefore $x_{1002}^2 + y_{1002}^2 \equiv 0$ and, by Section 2, the motion will be stable provided that the inequality (2.11) holds for $\omega_1 = 2\omega_2$. According to [7], the inequality (2.11) breaks down st two points α and β of the parametric plane, where $\alpha = C/A$ and $\beta = r_0/\omega_0$ (A and C are, respectively, the equatorial and polar moment of inertia of the satellite, r_0 denotes the component of the absolute angular velocity projected on the axis of symmetry and being the integral of motion and ω_0 is the angular velocity of the center of gravity moving along the orbit). When investigating the stability for the case

$\omega_1 = 3\omega_2$, we compute the values of the function $a_0 - |a_2|$ along the curve $\omega_1 = 3\omega_2$, beginning with the values of β of large absolute magnitude ($\beta = -114$). As a result we find, that the function is positive for $-1.743 < \beta < -1.566$ and $0.384 < \beta < 0.45$. Consequently, by Section 3, the motion of the type (1) is unstable over the above values of β. On the remainder of the resonance curve the stability can only be proved by considering the Hamiltonian (1.2) truncated beyond the fourth order terms.

Computations show that in the case of motion of the type (3), we have the instability along the whole resonance curve for both, $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$.

5. Numerical formulas . We know (see e.g. [8]) that a real, linear, canonical transformation exists, which reduces H_2 to the form corresponding to the normal oscillations. Therefore we can assume that the Hamiltonian (1.2) has the form

$$H = {}^1/_2 (p_1^2 + \omega_1^2 q_1^2) - {}^1/_2 (p_2^2 + \omega_2^2 q_2^2) + \sum_{\nu=3} g_{\nu_1\nu_2\nu_3\nu_4} q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4} +$$
$$+ \sum_{\nu=4} h_{\nu_1\nu_2\nu_3\nu_4} q_1^{\nu_1} q_2^{\nu_2} p_1^{\nu_3} p_2^{\nu_4} + \cdots \qquad (5.1)$$

Employing the canonical change of the variables

$$q_1 = {}^1/_2 q_1' + i\omega_1^{-1} p_1', \qquad p_1 = {}^i/_2 \omega_1 q_1' + p_1',$$
$$q_2 = -{}^i/_2 q_2' + \omega_2^{-1} p_2', \qquad p_2 = -{}^1/_2 \omega_2 q_2' + i p_2' \qquad (5.2)$$

we can reduce the Hamiltonian (5.1) to the form suitable for the Birkhoff transformation. The resulting Hamiltonian will have the form (2.1), where

$$g'_{0030} = x_{0030} + iy_{0030}, \quad g'_{3000} = -{}^1/_8 \omega_1^3 (y_{0030} + ix_{0030}), \quad g'_{1020} = x_{1020} + iy_{1020}$$
$$g'_{2010} = -{}^1/_2 \omega_1 (y_{1020} + ix_{1020}), \quad g'_{0120} = x_{0120} + iy_{0120}$$
$$g'_{2001} = {}^1/_2 \omega_1^2 \omega_2^{-1} (y_{0120} + ix_{0120}), \quad g'_{1011} = x_{1011} + iy_{1011}, \quad g'_{1110} = {}^1/_2 \omega_2 (y_{1011} + ix_{1011})$$
$$g'_{0021} = x_{0021} + iy_{0021}, \quad g'_{2100} = {}^1/_8 \omega_1^2 \omega_2 (y_{0021} + ix_{0021}), \quad g'_{1002} = x_{1002} + iy_{1002}$$
$$g'_{0210} = -{}^1/_2 \omega_1^{-1} \omega_2^2 (y_{1002} + ix_{1002}), \quad g'_{0012} = x_{0012} + iy_{0012} \qquad (5.3)$$
$$g'_{1200} = -{}^1/_8 \omega_1 \omega_2^2 (y_{0012} + ix_{0012}), \quad g'_{0111} = x_{0111} + iy_{0111}, \quad g'_{1101} = -{}^1/_2 \omega_1 (y_{0111} + ix_{0111})$$
$$g'_{0201} = x_{0201} + iy_{0201}, \quad g'_{0102} = 2\omega_2^{-1} (y_{0201} + ix_{0201}), \quad g'_{0003} = x_{0003} + iy_{0003}$$
$$g'_{0300} = {}^1/_8 \omega_2^3 (y_{0003} + ix_{0003})$$

$$x_{0030} = g_{0030} - \omega_1^{-2} g_{2010}, \quad y_{0030} = \omega_1^{-1} g_{1020} - \omega_1^{-3} g_{3000}, \quad x_{1020} = -{}^1/_2 g_{1020} - {}^3/_2 \omega_1^{-2} g_{3000}$$
$$y_{1020} = {}^3/_2 \omega_1 g_{0030} + {}^1/_2 \omega_1^{-1} g_{2010}, \quad x_{0120} = -{}^1/_2 \omega_2 g_{0021} + {}^1/_2 \omega_1^{-1} g_{1110} + {}^1/_2 \omega_1^{-2} \omega_2 g_{2001}$$
$$y_{0120} = -{}^1/_2 g_{0120} - {}^1/_2 \omega_1^{-1} \omega_2 g_{1011} + {}^1/_2 \omega_1^{-2} g_{2100}, \quad x_{1011} = -\omega_1 g_{0021} - \omega_1^{-1} g_{2001}$$
$$y_{1011} = \omega_1 \omega_2^{-1} g_{0120} + \omega_1^{-1} \omega_2^{-1} g_{2100}, \quad x_{0021} = \omega_2^{-1} g_{0120} - \omega_1^{-1} g_{1011} - \omega_1^{-2} \omega_2^{-1} g_{2100} \qquad (5.4)$$
$$y_{0021} = g_{0021} + \omega_1^{-1} \omega_2^{-1} g_{1110} - \omega_1^{-2} g_{2001}, \quad x_{1002} = -{}^1/_2 \omega_1 \omega_2^{-1} g_{0111} - {}^1/_2 g_{1002} + {}^1/_2 \omega_2^{-2} g_{1200}$$
$$y_{1002} = -{}^1/_2 \omega_1 g_{0012} + {}^1/_2 \omega_1 \omega_2^{-2} g_{0210} + {}^1/_2 \omega_2^{-1} g_{1101}, \quad x_{0012} = -g_{0012} + \omega_2^{-2} g_{0210} - \omega_1^{-1} \omega_2^{-1} g_{1101}$$
$$y_{0012} = \omega_2^{-1} g_{0111} - \omega_1^{-1} g_{1002} + \omega_1^{-1} \omega_2^{-2} g_{1200}, \quad x_{0111} = \omega_1^{-1} \omega_2 g_{1002} + \omega_1^{-1} \omega_2^{-1} g_{1200}$$
$$y_{0111} = -\omega_2 g_{0012} - \omega_2^{-1} g_{0210}, \quad x_{0201} = -{}^1/_4 \omega_2 g_{0102} - {}^3/_4 \omega_2^{-1} g_{0300}, \quad y_{0201} = {}^3/_4 \omega_2^2 g_{0003} + {}^1/_4 g_{0201}$$
$$x_{0003} = -\omega_2^{-1} g_{0102} + \omega_2^{-3} g_{0300}, \quad y_{0003} = -g_{0003} + \omega_2^{-2} g_{0201}$$

The coefficients accompanying the fourth degree terms in (2.1) which are relevant to our investigation, will be

$$h'_{2020} = -{}^3/_2\,\omega_1^2 h_{0040} - {}^3/_2\,\omega_1^{-2} h_{4000} - {}^1/_2\, h_{2020}$$
$$h'_{1111} = \omega_1\omega_2 h_{0022} + \omega_1^{-1}\omega_2^{-1} h_{2200} + \omega_1\omega_2^{-1} h_{0220} + \omega_2\omega_1^{-1} h_{2002}$$
$$h'_{0202} = -{}^3/_2\,\omega_2^2 h_{0004} - {}^3/_2\,\omega_2^{-2} h_{0400} - {}^1/_2\, h_{0202} \tag{5.5}$$
$$h'_{1003} = u_{1003} ++ iv_{1003}, \quad h_{2310} = -{}^1/_4\,\omega_1^{-1}\omega_2^3 (u_{1003} - iv_{1003})$$

where

$$u_{1003} = {}^1/_2\,\omega_1 h_{0013} + {}^1/_2\,\omega_2^{-2} h_{1300} - {}^1/_2\,\omega_2^{-1} h_{1102} - {}^1/_2\,\omega_1\omega_2^{-2} h_{0211}$$
$$v_{1003} = -{}^1/_2\,\omega_1\omega_2^{-1} h_{0112} - {}^1/_2\, h_{1003} + {}^1/_2\,\omega_2^{-2} h_{1201} + {}^1/_2\,\omega_1\omega_2^{-3} h_{0310} \tag{5.6}$$

Formulas (5.3) – (5.6) yield, for $\omega_1 = 3\omega_2$

$$l_{2020} = h'_{2020} + {}^{27}/_8\,\omega_2^2 (x^2_{0030} + y^2_{0030}) + {}^3/_2 (x^2_{1020} + y^2_{1020}) + {}^9/_{10} (x^2_{0120} + y^2_{0120}) - {}- {}^1/_2 (x^2_{1011} + y^2_{1011}) - {}^9/_{56}\,\omega_2^2 (x^2_{0021} + y^2_{0021}) \tag{5.7}$$
$$l_{1111} = h'_{1111} - {}^2/_3 (x^2_{1002} + y^2_{1002}) + {}^3/_{10}\,\omega_2^2 (x^2_{0012} + y^2_{0012}) - {}^9/_{14}\,\omega_2^2 (x^2_{0021} + y^2_{0021}) - {}- {}^{18}/_5 (x^2_{0120} + y^2_{0120}) + 2 (x_{0111}x_{1020} + y_{0111}y_{1020}) - 4\omega_2^{-1} (x_{0201}y_{1011} + x_{1011}y_{0201})$$
$$l_{0202} = h'_{0202} - {}^3/_8\,\omega_2^2 (x^2_{0003} + y^2_{0003}) - 6\omega_2^{-2} (x^2_{0201} + y^2_{0201}) + {}^1/_6 (x^2_{1002} + y^2_{1002}) + {}+ {}^1/_2 (x^2_{0111} + y^2_{0111}) + {}^3/_{40}\,\omega_2^2 (x^2_{0012} + y^2_{0012})$$
$$l_{1003} = x_{1003} + iy_{1003}, \quad l_{0310} = -{}^1/_{12}\,\omega_2^3 (x_{1003} -- iy_{1003})$$

where

$$x_{1003} = u_{1003} - {}^9/_5 (x_{0120} x_{0012} + y_{0120}y_{0012}) - \omega_2^{-1} (x_{1002}y_{1011} + x_{1011}y_{1002}) + {}+ 4\omega_2^{-2} (x_{1002}x_{0201} + y_{1002}y_{0201}) + {}^3/_2 (x_{0003} x_{0111} + y_{0003}y_{0111})$$
$$y_{1003} = v_{1003} - {}^9/_5 (x_{0120}y_{0012} - x_{0012}y_{0120}) - \omega_2^{-1} (y_{1011}y_{1002} - x_{1011}x_{1002}) + {}+ 4\omega_2^{-2} (x_{0201}y_{1002} - x_{1002}y_{0201}) + {}^3/_2 (x_{0111} y_{0003} - x_{0003}y_{0111})$$

In the practical investigations of stability, the Hamiltonian should be reduced to the form (5.1) and formulas given in Section 5 employed, together with the Theorems 2.1 and 3.1.

The author thanks V. A. Sarychev for constructive criticism.

BIBLIOGRAPHY

1. Malkin, I. G., Theory of Stability of Motion. 2nd ed., M.-L., "Nauka", 1966.
2. Arnol'd, V. I., Small denominators and the problems of stability of motion in the classical and celestial mechanics. Usp. matem. nauk, Vol. 18, №6, 1963.
3. Whittaker, E. T., Analytical Dynamics. CUP, 1959.
4. Birkhoff, G. D., Dynamic Systems. Am. Math. Soc. rev. ed., 1966.
5. Chetaev, N. G., Stability of Motion. 3rd ed., "Nauk^a , 1965.
6. Beletskii, V. V., Motion of an Artificial Satellite Relative to the Center of Gravity. M., "Nauka", 1965.
7. Markeev, A. P., Resonance effects and the stability of the steady rotations of a satellite. Kosmich. issled., Vol. 5, №3, 1967.
8. Bulgakov, B. V., Normal coordinates. PMM Vol. 10, №2, 1946.

Translated by L. K.